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OF THE TWO-PHOTON DECAY

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## SPACE-TIME DESCRIPTION OF THE TWO-PHOTON DECAY

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#### ABSTRACT

The time correlation of photons in a two-photon decay is shown to depend on the instantaneous nature of the wave-function collapse in an essential way so the latter hypothesis can be verified by the experimental study of these correlations.

#### АННОТАЦИЯ

Показано, что временная корреляция фотонов в двухфотонном распаде существенно зависит от того предположения, что редукция волновой функции имеет мгновенный характер. Поэтому такое предположение можно проверить путем экспериментального изучения этих корреляций.

#### KIVONAT

Megmutatjuk, hogy a kétfotonos bomlásban a fotonok időbeli korrelációja lényegesen függ attól a feltevéstől, hogy a hullámfüggvény redukciója pillanatszerű. A korrelációk mérése útján tehát ellenőrizni lehetne ezt a feltevést.



## 1. INTRODUCTION

Two-photon decay is a special class of forbidden electromagnetic transition which take place when both the ground state  $\phi_a$  and the first excited state  $\phi_b$  of the atom are of zero angular momentum. In this case the decay of the first excited state through the emission of a single photon is forbidden by angular momentum conservation but the de-excitation through two-photon emission is allowed. In a perturbation treatment this second order process appears as a result of the virtual excitation of some states  $\phi_c$ , lying above the decaying state  $\phi_b$ . The total energy of the photons in the final state is, nevertheless, equal to the excitation energy  $\hbar\omega_{ba}$  of the state  $\phi_b$  but the single-photon energy spectrum is continuous. It was calculated by M. Goeppert-Mayer [1] as early as in 1929 for astrophysical purposes. Beside this continuous energy distribution the second characteristic feature of the two-photon transitions is the time correlation of the photons as detected in a delayed coincidence experiment. The purpose of the present work is to study this time distribution in some detail.

At first sight the time correlation can present no problem at all. Assume that the photon detectors are situated at distances  $r_A$  and  $r_B$  respectively from the source. Since the intermediate states are virtual rather than real states the two photons are emitted in coincidence [2]. If, therefore,  $\tau$  denotes the /positive/ time delay between the responses of the detectors the relative time distribution as a function of  $\tau$  will consist of a single peak at  $\tau = \frac{1}{c}|r_A - r_B|$ . Since the width of the peak is of the order of  $\omega^{-1}$  where  $\omega$  denotes some frequency which is the same order of magnitude as the Bohr-frequencies  $\omega_{ba}, \omega_{ca}, \omega_{cb}$ , the peak can be considered as infinitely narrow.

This argumentation, however, can be criticized from several points of view. It strongly relies on the concept of the emission event whose reality can be seriously doubted. Indeed, it is well known that in a single photon decay  $X^* \rightarrow X + \gamma$  the reality of the emission event depends on the circumstances of the observation [3]. For example, using Michelson-Morley type interferometer with arms of unequal length, one can study the interference between those parts of the wave which were emitted at different moments of time. The existence of an interference pattern shows clearly that no definite emission event can be attributed to the decay process. When, on the other hand, the



time correlation between the moment of appearance of the ground state X and the moment of detection of the photon is observed the emission event is a completely real event which can be identified with the moment of observation of the ground state X.

Since the time correlation measurement in a two-photon decay has more resemblance on the second example as on the first one we are inclined to expect that the time correlation between the photons can in fact be interpreted in terms of a real emission event. Only a full quantum mechanical treatment can decide to what extent is this expectation valid.

The second characteristic moment of the argumentation concerns the explanation of the simultaneity of the photons. It is based on the picture suggested by the structure of the second order term of the perturbation series, according to which the process is to be considered as a virtual cascade  $b+c + \gamma \rightarrow a + 2\gamma$ . Since the excitation energy of the intermediate state exceeds the energy of the initial state by an amount of  $\hbar\omega$  the time delay between the photons cannot exceed  $\omega^{-1}$  in consequence of the time-energy uncertainty relation.

However, the real content of the time-energy uncertainty relation is, perhaps, the existence of "some correlation between energy dispersion and time variation of dynamical variables [4]". In a true stationary state the expectation value of any of the dynamical variables is independent of time, therefore, in a quasistationary state the time rate of change of the dynamical variables is proportional to the decay constant  $\gamma$ . Since the line-width  $\hbar\gamma$  is much smaller than  $\hbar\omega$  the decay wave-function is clearly incompatible with such a rapid process which is assumed in the picture of the virtual cascade.

Actually this picture can be replaced by another one which corresponds equally well to the formula of the second order perturbation calculation but does not involve such enormously large time rates. According to this second picture, the decaying state  $\phi_b$  has to be considered together with its photon cloud. Then, to first order, the decaying state will be a superposition of a large component  $\phi_b$  and a small component  $\int d\omega \phi_c \phi_\omega$  where  $\phi_\omega$  is a single-photon state. To this order this superposition is a genuine stationary state. It, however, becomes unstable in the next order since the small component has nonzero matrix element to the final state  $\int d\omega d\omega' \phi_a \phi_{\omega\omega'}$  where  $\phi_{\omega\omega'}$  is a two-photon state. Hence only one of the photons of the two-photon decay derives from a current as a source the other photon bears its origin from the vacuum fluctuations whose spectrum undergo modifications during the radiation of the current. From this point of view the generation of this photon resembles a parametric process known from nonlinear optics [5] and black-hole radiation [6].

The third point to be discussed is connected with the propagation of the photons from the source to the detectors. As it is well known [7] in a quantum mechanical treatment of time distributions the atoms of the detectors have to be included into the system explicitly. The reason is that the distribution



of the moments of time cannot be obtained from the wave functions by projecting them on the eigenfunctions of a time operator since time is a parameter, having no operator representation. The inclusion of the detector atoms permits us to replace this projection by a projection over the eigenstates of their Hamiltonian. On the language of diagrams the photons are, therefore, inner lines and must be represented by propagators. Since the distances and time intervals involved are macroscopic ones the photons are real rather than virtual, i.e. only the singularities of the propagator contribute significantly to the amplitude. But different propagators have different singularity structure and the answer depends on the propagator used to describe the photon.

Obviously the choice of the propagator is not a matter of taste but is determined by the rules of quantum mechanics and the boundary conditions. If one accepts the usual Feynman-rules as the correct prescriptions, i.e. chooses the causal propagator to describe the photon one immediately runs into a serious problem. The causal propagator is singular on both the positive and negative light cone and describes retarded as well as advanced propagation. In the early days of covariant quantum electrodynamics Fierz showed [8] why in spite of this singularity structure we do not come into conflict with experience which shows up only retarded effects. The explanation of Fierz was based on the observation that in the interaction picture the current of systems, emitting photons, is of positive frequency while that of systems which absorb photons is of negative frequency. This strong correlation between the sign of the frequency of the current and the emitting or absorbing nature of the process is sufficient to show that under macroscopic circumstances only that singularity contributes to the amplitude which is retarded with respect to an emitter and advanced with respect to an absorber.

However, while the argumentation of Fierz is certainly valid for single-photon processes it fails for a two-photon emitter. In the virtual cascade picture the current which corresponds to the first step of the virtual cascade is of negative frequency and, according to the derivation of Fierz, it enhances the light cone which is advanced with respect to itself. The result is that, assuming a real emission event, the peak in the relative time distribution appears at  $\tau = \frac{1}{c}|r_A + r_B|$  instead of  $\tau = \frac{1}{c}|r_A - r_B|$  which is a highly paradoxical result. The replacement of the causal propagators by retarded ones would cure the trouble but then it remains to show that for the problem under consideration the retarded propagators are those which actually meet the requirements of the theory. The discussion in the subsequent sections will lead us to conclude that despite of its paradoxical nature the relative time distribution does indeed contain a peak at  $\tau = \frac{1}{c}|r_A + r_B|$  in addition to that at  $\tau = \frac{1}{c}|r_A - r_B|$ .

The last problem which requires special attention is the reduction of the wave-function at the moment of detection of the first photon since the relative time distribution is strongly influenced by the time development of this reduced wave-function. It seems unlikely that there can be drawn any



conclusion on the coincidence spectrum from a consideration which concentrates on the emission process alone and ignores completely the influence of the detectors.

Time distributions of photons under stationary circumstances have been intensively studied in connection with the structure of the light beams [7]. We take over from these considerations the so called counting rate formula which is the starting point of this type of calculations. This formula has no such deep foundation as the rule for calculating probability distributions of observables which are represented by Hermitean operators but it proved itself to work well in the studies of the light beams and, which is more important, quantum theory in its present-day form does not provide us any other more direct prescription to calculate distributions of moments of time which are "chosen" by the physical system itself.

The calculation leads to rather complicated expressions already at the first stages. In order to simplify them we confine ourselves to a single intermediate state  $\varphi_c$ , take the atom infinitely heavy and replace the electromagnetic field by a zero mass scalar field the quanta of which we continue to call photons. System of units in which  $\hbar = c = 1$  will be employed. Since  $r_A$  and  $r_B$  are macroscopic distances terms of the relative order  $(\omega r_A)^{-1}, (\omega r_B)^{-1}$  will be systematically neglected. We will neglect terms of the relative order  $(\omega(t - r_A))^{-1}, (\omega(t - r_B))^{-1}, (\omega t)^{-1}$  as well where  $t$  is the moment of the first detector response. This assumption means that we confine ourselves to such region of space-time which is far away from the light cone of the state preparation event the neighborhood of which is strongly influenced by the largely uncontrollable peculiarities of the state preparation.

The aim of the paper is to give a qualitative picture of the time correlations. No attempt will be made to obtain elaborated formulae suitable for numerical estimates.

## 2. THE WAVE-FUNCTION OF THE DECAYING STATE

Let us denote the eigenfunctions of the source Hamiltonian by  $\varphi_a, \varphi_b$  and  $\varphi_c$ . The scalar field  $\phi(\underline{r})$  will be coupled to the source through a dipole-like interaction so that the total Hamiltonian is given by the expression

$$H = \frac{1}{2} \int d^3x [\pi^2(\underline{r}) + (\nabla \phi(\underline{r}))^2] + H_{\text{source}} + Q\phi(\underline{r} = 0) \quad (1)$$

where  $Q$  is an operator in the Hilbert-space of the source. In order to ensure that  $\varphi_b$  can decay only to two-photon final states we assume that the matrix element of  $Q$  between  $\varphi_b$  and  $\varphi_a$  is equal to zero:  $Q_{ba} = 0$ . If we prescribe zero angular momentum to all the three states of the source then only the s-wave of the field will play role. Confining ourselves to this partial wave, the expansion of the Schrödinger-operator  $\phi(\underline{r})$  in terms of creation and annihilation operators is of the form



$$\phi(r) = \int_0^{\infty} d\omega [a(\omega)u(\omega|r) + a^+(\omega)u^*(\omega|r)] = \phi^{(+)}(r) + \phi^{(-)}(r)$$

where

$$u(\omega|r) = \frac{1}{2\pi} \sqrt{\omega} j_0(\omega r) = \frac{1}{2\pi} \sqrt{\omega} \frac{\sin \omega r}{\omega r}$$

and

$$[a(\omega), a^+(\omega')] = \delta(\omega - \omega')$$

To first order the wave-function of the first excited state is

$$\phi'_b = \phi_b \phi_{vac} - \int_0^{\infty} d\omega A_{\omega} \phi_{\omega} \phi_c \quad (2)$$

where  $\phi_{vac}$  is the vacuum of the scalar field  $a(\omega)\phi_{vac}=0$ ,  $\phi_{\omega}=a^+(\omega)\phi_{vac}$  and\*

$$A_{\omega} = - \frac{iQ_{cb}}{2\pi} \frac{\sqrt{\omega}}{\omega_{cb} + \omega}.$$

In the next order  $\phi'_b$  becomes unstable. To this order the wave-function valid for times of the order of the life-time can be obtained from  $\phi'_b$  by means of a standard Wigner-Weisskopf perturbation calculation [9], leading to

$$\begin{aligned} \Psi(t) = e^{-i(\omega_{ba} - \frac{1}{2}\gamma)t} & \left[ \phi_b \phi_{vac} - \int_0^{\infty} d\omega A_{\omega} \phi_{\omega} \phi_c \right] + \\ & + \int_0^{\infty} d\omega d\omega' A_{\omega\omega'}(t) e^{-i(\omega + \omega')t} \phi_{\omega\omega'} \phi_a, \end{aligned} \quad (3)$$

where

$$\phi_{\omega\omega'} = a^+(\omega) a^+(\omega') \phi_{vac},$$

and

$$\begin{aligned} A_{\omega\omega'}(t) = - \frac{Q_{ac} Q_{cb}}{(2\pi)^2} \sqrt{\omega\omega'} & \left( \frac{1}{\omega_{cb} + \omega} + \frac{1}{\omega_{cb} + \omega'} \right) \cdot \\ & \cdot \frac{1 - e^{-i(\omega_{ba} - \omega - \omega' - \frac{1}{2}\gamma)t}}{\omega_{ba} - \omega - \omega' - \frac{1}{2}\gamma} \end{aligned} \quad (4)$$

\*As usual, dipole approximation leads to divergent frequency integrals which require cutoff. For our purposes there is no need to introduce this cutoff explicitly.



This calculation provides us also the following formula for the decay constant:

$$\begin{aligned} \gamma &= 2\pi \int_0^\infty d\omega d\omega' |(\varphi_{\omega\omega}, \varphi_a, H_{int} \varphi'_b)|^2 \delta(\omega_{ba} - \omega - \omega') = \\ &= \frac{|Q_{ac} Q_{cb}|^2}{(2\pi)^3} \int_0^\infty d\omega d\omega' \delta(\omega_{ba} - \omega - \omega') \cdot \omega \omega' \left( \frac{1}{\omega_{cb} + \omega} + \frac{1}{\omega_{cb} + \omega'} \right)^2 \end{aligned}$$

which coincides with the corresponding expression of the second order perturbation theory.

What happens if at  $t = 0$  the wave-function is different from  $\varphi'_b$ ? As usual it has to be expanded in terms of the quasi-stationary states which will decay independently if the level spacings are large. In the special case when the initial wave-function is just  $\varphi_b \varphi_{vac}$  this expansion takes the simple form\*

$$\varphi_b \varphi_{vac} = \varphi'_b + \int_0^\infty d\omega A_\omega \varphi_\omega \varphi_c$$

as it follows directly from (2) and the two terms decay independently. In the small part of the events determined by the quantity  $|A_\omega|^2$  an instantaneous photon appears which is followed by the decay of  $\varphi_c$  either to the ground state  $\varphi_a$  or to the first excited state  $\varphi_b$ . Since these transitions are allowed, the decay constant  $\gamma_c$  of  $\varphi_c$  is much larger than  $\gamma$ . Therefore, after a short initial time interval only the first component remains and the process continues to proceed in the same way as if the initial states was  $\varphi'_b$ . Obviously the same thing happens for an arbitrary initial state of a real atom also.

### 3. THE COUNTING RATE FORMULA

Let  $W(t, \tau)$  denote the probability that the first photon is detected at the momentum  $t$  while the second photon is observed at a time  $\tau$  later. The analysis of the photon detection process leads to a calculational formula known as the counting rate formula [7] which in the form adapted to our problem and after emission of irrelevant constant factors is given as

$$\omega(t, \tau) = |M(\underline{r}_1, \underline{r}_2; t, \tau)|^2$$

with

$$M(\underline{r}_1, \underline{r}_2; t, \tau) = (\varphi_a \varphi_{vac}, \phi_H^{(+)}(t+\tau, \underline{r}_2) \phi_H^{(+)}(t, \underline{r}_1) \varphi'_b). \quad (5)$$

\*Using the resolvent method [5], it is possible to show that when  $Q_{ba} = 0$  the single unperturbed energy eigenstate  $\varphi_b$  gives rise to two states, decaying exponentially with different decay constants.



Here  $\phi_a \phi_{vac}$  represents the ground state of  $H$  to the order in  $Q$  which is required by the accuracy of the calculation,  $\phi_H^{(+)}(t, \underline{r})$  are Heisenberg operators which at  $t = 0$  coincide with the Schrödinger operators  $\phi^{(+)}(\underline{r})$  and  $\underline{r}_1$  ( $\underline{r}_2$ ) is the coordinate  $\underline{r}_A$  or  $\underline{r}_B$  of that detector which operated earlier /later/.

This formula which is nothing but the second time derivative of the usual perturbative expression for the probability to find the detectors excited was obtained under the following assumptions:

a/ A macroscopic detector is a collection of independent atomic detectors, the coordinates  $\underline{r}_A$  and  $\underline{r}_B$  may be identified with the coordinate of any of them.

b/ If a detector at some moment of time is found in the excited state  $\phi_d$  the wave-function of the whole system is projected onto the subspace, containing  $\phi_d$ , and this projected wave-function serves as the initial condition for further causal development of the state.

c/ From the Hamiltonian, governing this further time development, the interaction of the above mentioned detector with the field is omitted. This step serves to simulate the irreversible nature of the detector response.

d/ The detectors are uniformly sensitive over the whole frequency range, including its negative part. If this assumption is abandoned the calculational formula becomes sufficiently more complicated. At the same time the sensitivity of the detectors at negative frequencies causes no trouble if the radiation field consists of positive frequency components alone. With respect to the variable  $t$  this is indeed the case as it can be seen explicitly from (3) and the subsequent consideration will show that the situation is similar with respect to  $\tau$  too.

As a consequence of the Heisenberg equations of motion for  $\phi_H^{(+)}$   $M$  satisfies the wave equation

$$\left(\frac{\partial^2}{\partial \tau^2} - \nabla^2\right)M(\underline{r}_1, \underline{r}_2; t, \tau) = -\frac{1}{2}\delta(\underline{r}_2)\chi(\underline{r}_1; t, \tau) \quad (6)$$

where

$$\chi(\underline{r}_1; t, \tau) = (\phi_a \phi_{vac}, Q_H(t+\tau)\phi_H^{(+)}(t, \underline{r}_1)\phi_b').$$

The solution of (6) has to be subjected to the following boundary conditions:

- i/ at  $\tau=0$   $M$  coincides with the given function  $M(\underline{r}_1, \underline{r}_2; t, 0)$  and
- ii/ as  $\tau \rightarrow \infty$   $M$  is of purely positive frequency.

The first condition is obvious while the second one can be ascertained by inserting the system of eigenstates of  $H$  into  $M$  and noticing that neither of these states has energy smaller than that of  $\phi_a \phi_{vac}$ . Since  $M$  is defined only for  $\tau > 0$  its frequency content is in principle definite only in the limit  $\tau \rightarrow \infty$ . If, however,  $\tau$  is not microscopically small the frequency is practical-



ly positive over  $\tau$  intervals, containing many periods, and this is sufficient for the applicability of the counting rate formula in the form of (5).

The solution of (6), satisfying the above boundary conditions, can be written as

$$M(\underline{r}_1, \underline{r}_2; t, \tau) = \frac{1}{2} \int_0^\infty d\tau' [D_F(\tau - \tau', \underline{r}_2) - iD^+(\tau + \tau', \underline{r}_2)] \chi(\underline{r}_1; t, \tau') + \\ + 2i \int d^3 \underline{r}'_2 \frac{\partial D^+(\tau, \underline{r}_2 - \underline{r}'_2)}{\partial \tau} M(\underline{r}_1, \underline{r}'_2; t, 0) \quad (7)$$

where

$$D_F(x) = i\theta(x^0)D^+(x) + i\theta(-x^0)D^-(x)$$

is the Feynman-propagator and

$$D^\pm(x) = \int \frac{d^3 k}{(2\pi)^3 2\omega} e^{\pm i k x} \quad (k^0 = \omega = |\underline{k}|)$$

are standard solutions of the homogeneous equation.

In the Schrödinger picture the function  $\chi$  can be expressed as

$$\chi(\underline{r}_1; t, \tau) = (\varphi_a \varphi_{\text{vac}}, e^{iH(t+\tau)} Q e^{-iH\tau} \varphi^{(+)}(\underline{r}_1) \Psi(t))$$

where

$$\Psi(t) = e^{-iHt} \varphi'_b \quad (8)$$

is the Wigner-Weisskopf wave-function (3). To second order in  $Q$  the only contributing term of  $\Psi(t)$  is that, containing  $A_\omega$ , and we have

$$\chi(\underline{r}_1; t, \tau) = e^{-i(\omega_{ba} - \frac{i}{2}\gamma)t} \cdot \frac{iQ_{ac}Q_{cb}}{2\pi} \cdot \\ \cdot \frac{i}{2\pi r_1} e^{-i(\omega_{ca} - \frac{i}{2}\gamma_c)\tau} \cdot \int_0^\infty d\omega \frac{\sin \omega r_1}{\omega_{cb} + \omega} \quad (9)$$

Since this function vanishes as  $\tau \rightarrow \infty$  the second of the boundary conditions is satisfied by the solution (7). The first condition is also satisfied because

$$2i \left[ \frac{\partial D^+(\tau, \underline{r}_2 - \underline{r}'_2)}{\partial \tau} \right]_{\tau=0} = \delta(\underline{r}_2 - \underline{r}'_2)$$

and the first term of (7) vanishes at  $\tau=0$ .



Actually this term can be dropped since the leading  $r_1$  dependence of  $M$  is  $r_1^{-1}$  while, as a consequence of the Riemann-Lebesgue lemma, the contribution of (9) goes to zero faster than  $r_1^{-1}$ . If in the retained second term of (7) we go over to polar coordinates we obtain

$$M(r_1, r_2; t, \tau) = \frac{1}{r_1 r_2} \int_0^\infty dr'_2 G(r_2, r'_2; \tau) \cdot \left[ r_1 r'_2 M(r_1, r'_2; t, 0) \right] \quad (10)$$

where

$$G(r_2, r'_2; \tau) = \frac{2}{\pi} \int_0^\infty d\omega e^{-i\omega\tau} \sin\omega r_2 \cdot \sin\omega r'_2 = \quad (11)$$

$$= \frac{1}{2\pi i} \left[ \frac{1}{r'_2 - r_2 + \tau - i\epsilon} + \frac{1}{r'_2 + r_2 - \tau + i\epsilon} - \frac{1}{r'_2 + r_2 + \tau} - \frac{1}{r'_2 - r_2 - \tau + i\epsilon} \right]$$

#### 4. SPACE CORRELATION OF PHOTONS IN THE WAVE-FUNCTION OF THE DECAYING STATE

In order to calculate  $M(r_1, r_2; t, 0)$  we put in (5)  $\tau$  equal to zero and go over to Schrödinger-picture. Using (8) and (3) we obtain the expression

$$M(r_1, r_2; t, 0) = \int_0^\infty d\omega d\omega' A_{\omega\omega'}(t) e^{-i(\omega+\omega')t} (\phi_{vac}, \phi^{(+)}(r_2) \phi^{(+)}(r_1) \hat{a}^{(+)}(\omega) \hat{a}^{(+)}(\omega') \phi_{vac}) =$$

$$= - \frac{2}{(2\pi)^2 r_1 r_2} \int_0^\infty \frac{d\omega d\omega'}{\sqrt{\omega\omega'}} A_{\omega\omega'}(t) \sin\omega r_1 \cdot \sin\omega' r_2 \cdot e^{-i(\omega+\omega')t}.$$

Introducing here  $A_{\omega\omega'}(t)$  from (4) and changing the integration variables from  $\omega, \omega'$  to  $u=\omega+\omega', v=\omega-\omega'$ , we obtain after a few steps

$$M(r_1, r_2; t, 0) = \frac{Q_{ac} Q_{cb}}{2(2\pi)^4 r_1 r_2} e^{i(r_1 - r_2)\omega_{cb}} \int_0^\infty \frac{du}{u - \omega_{ba} + \frac{i}{2}\gamma} \cdot \left[ e^{-i(t-r_1)u} + \right.$$

$$+ e^{-i(t+r_2)u} e^{-i(\omega_{ba} - \frac{i}{2}\gamma)t} \cdot (e^{ir_1 u} + e^{-ir_2 u}) \left. \right]. \quad (12)$$

$$\cdot [Ei(i(r_2 - r_1)(\omega_{cb} + u)) - Ei(i(r_2 - r_1)\omega_{cb})] + (r_1 \leftrightarrow r_2)$$

\*In the case of a real cascade the source term gives important contribution.



Although no photon position operator exists [10] it is convenient to introduce formally photon position eigenstates  $\varphi_r$  and to consider the two-photon state

$$\int dr_1 dr_2 M(r_1, r_2; t, 0) \varphi_{r_1} \varphi_{r_2} \quad (13)$$

since, in the context of the counting rate formula, the space distribution of photons is proportional to  $|M(r_1, r_2; t, 0)|^2$  as if (13) were an expansion in terms of genuine eigenstates.

To see the space distribution of the photons we specify  $M(r_1, r_2; t, 0)$  to the interval  $R_1 < r_2 < R_2$ , where  $R_1$  is between the origin and  $r_1$  ( $\omega R_1 \gg 1, \omega(r_1 - R_1) \gg 1$ ) and  $R_2$  is between  $r_1$  and  $t$  ( $\omega(R_2 - r_1) \gg 1, \omega(t - R_2) \gg 1$ ). If, in addition, we use the inequalities  $\omega r_1 \gg 1, \omega(t - r_2) \gg 1, \omega t \gg 1$  for the fixed\* parameters  $r_1$  and  $t$  discussed at the end of the Introduction we can put (12) into the form /see Appendix/

$$\begin{aligned} M(r_1, r_2; t, 0) = & (-2\pi i) \frac{Q_{ac} Q_{cb}}{2(2\pi)^4 r_1 r_2} e^{-i(\omega_{ba} - \frac{1}{2}\gamma)t} \cdot \\ & \cdot \{ e^{i(r_1 - r_2)\omega_{cb}} e^{ir_1(\omega_{ba} - \frac{1}{2}\gamma)} \left[ \text{Ei}(i(r_2 - r_1)(\omega_{ca} - \frac{1}{2}\gamma)) - \right. \\ & \left. - \text{Ei}(i(r_2 - r_1)\omega_{cb}) \right] + (r_1 \leftrightarrow r_2) \} \end{aligned} \quad (14)$$

From this expression the following space distribution can be obtained:

$$\begin{aligned} |M(r_1, r_2; t, 0)|^2 = & |(-2\pi i) \frac{Q_{ac} Q_{cb}}{2(2\pi)^4 r_1 r_2}|^2 e^{-\gamma(t-R)} \cdot \\ & \left\{ \begin{aligned} & \left( \frac{\xi^2 - 1}{\xi} \right)^2 \left[ \left( \frac{\sin \omega_{ba} \Delta r}{\omega_{ba} \Delta r} \right)^2 + \frac{\gamma^2}{4\omega_{ba}^2} \left( \frac{\text{sh} \frac{\gamma}{2} \Delta r}{\frac{\gamma}{2} \Delta r} \right)^2 \right] & \text{if } \omega \Delta r \rightarrow \infty \\ & 4 \ln^2 \xi & \text{if } \Delta r = 0 \end{aligned} \right. \end{aligned}$$

where  $R = \frac{1}{2}(r_1 + r_2), \Delta r = \frac{1}{2}(r_1 - r_2)$  and  $\xi = \frac{\omega_{ca}}{\omega_{ba}} \geq 1$ .

\*At the present stage  $r_2$  cannot yet be constrained by similar inequalities since in (10) we will integrate over it.



For the purpose of reference let us define  $|M'(r_1, r_2; t, 0)|^2$  by the upper line of this formula and put for the moment  $\gamma$  equal to zero.  $|M'(r_1, r_2; t, 0)|^2$  describes a strongly correlated radial space distribution of the photons whose effective correlation width is of the order of  $\omega_{ba}^{-1}$ . At  $\Delta r=0$  the true distribution  $|M(r_1, r_2; t, 0)|^2$  for  $\xi > 1$  is smaller than the reference distribution. Hence, the true effective width, being also of the order of  $\omega^{-1}$ , is larger than the width of the effective distribution. Since the latter is of the order of  $\omega_{ba}^{-1}$  the width of the true distribution does not go to zero with the increasing energy of the virtual state  $\phi_c$  as could have been expected on the basis of the time-energy uncertainty relation.

The last term, containing  $\gamma$ , leads to unexpected behaviour. Since  $(\frac{\text{sh}x}{x})^2$  is an increasing function it leads to a kind of anticorrelation in the space distribution. The origin of this term is clear: it reflects the fact that the probability distribution increases exponentially as we approach the wave-front and this growth finally dominates the decrease in inverse power. This increase is, of course, limited by the inequalities  $r_2 < R_2, r_1 < t$ . Because of these constraints and the smallness of  $\gamma/\omega_{ba}$  this term becomes appreciable only for very large times. Since in this limit the validity of the Wigner-Weisskopf wave-function is questionable [11] we will not discuss the possible effect of this term any more though our formulae will contain its influence.

## 5. THE PATTERN OF THE TIME CORRELATIONS

We see that the space distribution of the photons at a given time  $t$  does not differ very much from what can be expected on purely intuitive grounds [2]. If the first photon is observed at the macroscopic distance  $r_1$  from the origin the second photon will be distributed in a spherical shell of the mean radius  $r_1$  and the amplitude of this distribution will serve as the initial value for the further development of the single photon wave-function.

For the purpose of a first qualitative discussion assume that the spherical shell mentioned above is infinitely thin i.e. we approximate  $M(r_1, r_2; t, 0)$  by

$$M_0(r_1, r_2; t, 0) = \frac{1}{4\pi r_2} \delta(r_2 - r_1) \quad (15)$$

where an omitted proportionality factor may depend on  $r_1$  and  $t$ . If we put this expression into (10) and use (11) we see at once that (15) gives rise to two spherical waves, an outgoing and an ingoing wave. After having reached the origin the ingoing component continues to propagate as a second outgoing wave.

Let us assume for definiteness that  $r_A < r_B$ . If the first photon is observed by the detector at  $r_A$  then  $r_1 = r_A$  and  $r_2 = r_B$ . The outgoing component, starting at  $r_A$ , leads to coincidences at around  $\tau = r_B - r_A$ , while the coincidences due to the ingoing component are concentrated around  $\tau = r_A + r_B$ . If the detector at



$r_B$  operates earlier then only the ingoing component is effective but it alone gives two peaks at the same  $\tau$ . If we denote by  $t_A$  and  $t_B$  the moments of responses of the detectors then, since  $\tau = |t_B - t_A|$ , we will have in general four peaks as a function of  $t_B - t_A$  though only one of them /that around  $t_B - t_A = r_B - r_A > 0$ / was originally expected to occur.

Owing to the paradoxical consequences of the ingoing wave, one is inclined to suspect that this component is somehow spurious. It is indeed difficult to believe that the second photon which before the first detector response moved in the outward direction changed its direction of motion only because the other photon was suddenly removed.

This expectation is based on the experience in classical field theory where it would indeed be completely reasonable. If we were dealing with a classical problem we would have to fix both  $M_O(r_1, r_2; t, \tau)$  and  $\frac{\partial M_O(r_1, r_2; t, \tau)}{\partial \tau}$  on the boundary  $\tau=0$  and it is the normal derivative  $\frac{\partial M_O}{\partial \tau}$  which brings in additional information on the initial motion of the packet. If, for example, we implement (15) by the second boundary condition

$$\left. \frac{\partial M_O(r_1, r_2; t, \tau)}{\partial \tau} \right|_{\tau=0} = - \frac{1}{4\pi r_2} \frac{\partial}{\partial r_2} \delta(r_2 - r_1)$$

we obtain the solution

$$M_O(r_1, r_2; t, \tau) = \frac{1}{4\pi r} [\delta(r_2 - t_1 - \tau) - \delta(r_2 + r_1 + \tau)]$$

which at  $\tau \geq 0$  is purely outgoing. In the same way we can obtain purely ingoing solution too.

However, the boundary value problems associated with the classical and the quantum field theories are different. Instead of the condition on  $\frac{\partial M}{\partial \tau}$  at  $\tau=0$  in the quantum theory we have a constraint on the sign of the frequencies. This condition is not fulfilled by the above solution which is the commutator function  $D(r_2, \tau + r_1)$ . It is, therefore, impossible to dispense with the ingoing component though its relative importance depends on the functional form of  $M(r_1, r_2; t, 0)$ .

The condition for the effective suppression of the ingoing wave is that the Fourier-component  $\tilde{M}$  of  $M$  defined by the relation

$$M(r_1, r_2; t, 0) = \int_{-\infty}^{\infty} dk e^{ikr_2} \tilde{M}(r_1, k; t, 0)$$

be confined to large positive values of  $k$ . In this case the integration contour in (10) can be rotated into the positive imaginary axis and only the poles of  $G$  which are crossed give appreciable contribution. Since only the first term of the (11), corresponding to the outgoing wave, has its pole in the upper half plane the outgoing component will be enhanced.



As we have seen in the previous section  $|M|^2$  is a narrow distribution of the width  $\omega_{ba}^{-1}$ . Hence, its Fourier-components are important in the interval  $(-\omega_{ba}, \omega_{ba})$ . A factor  $e^{i\Omega r_2}$  in the amplitude  $M$  shifts this region by an amount  $\Omega$  toward positive values. Since the largest available  $\Omega$  is  $\omega_{ca}$  we see that an effective suppression of the ingoing component is possible only if the parameter  $\omega_{ca}/\omega_{ba}$  is sufficiently large i.e. for high excitation energy of the virtual state  $\phi_c$ .

Another interesting feature of the time correlation pattern arises as a consequence of the fact that at  $\tau=r_2$  the integral (10) diverges logarithmically on the lower limit. This divergence does not arise in the example treated above since  $r_2 M_0$  vanishes at  $r_2=0$  but for the true  $M$  of (12) it does appear /see Appendix/. Because of the strong correlation in  $M(r_1, r_2; t, 0)$  the singularity is of the relative order of  $(\omega r_1)^{-1}$ . Finite multiples of this quantity are neglected in our calculation but when multiplied by the divergent factor  $\ln|\tau-r_2|$  it has to be retained. The infinite value at  $\tau=r_2$  presents no problem since the singularity is integrable.

In order to explain the origin of this singularity we notice that in the course of the derivation of the counting rate formula our immediate goal is to obtain the distribution function from which  $w(t, \tau)$  follows by derivations. The reason for this procedure is that the form of the detector wave function at a prescribed moment of time  $t$  provides us only the probability that up to this moment a photon has already been detected. This probability is given by the weight of that part of the detector wave-function which describes excited components. It is the rate of growth of this quantity which determines the differential distribution function  $w$ .

At the moment  $t$  of the operation of the first detector the wave-function changes instantaneously. After  $t$  it contains an integral  $\int d\omega \phi_a \phi_\omega$  and a term  $\phi_c \phi_{vac}$  whose influence, however, was neglected by dropping the source term of (6). Hence, after  $t$  the source does not radiate any more.

The singularity of the field caused by this instantaneous switching off the source reaches the second detector at  $\tau=r_2$ . Confining ourselves to a domain around this moment which is outside the peaks of the ingoing and outgoing waves, this statement may also be formulated by saying that before the moment  $\tau=r_2$  the detector amplitude does not depend on  $t$ . After this moment it does not depend on the running time  $\tau$  since the source is switched off. Therefore, the second mixed derivative of the square of the detector amplitude will be singular\* at  $\tau=r_2$  which in terms of the variable  $t_B - t_A$  means two new peaks, one at  $+r_A$  and the other at  $-r_B$ . These peaks if observed would be rather direct indications of the instantaneous nature of the wave-function collapse.

\*An analogous argumentation proves that for the single-photon decay  $X \rightarrow X + \gamma$  in a time correlation experiment mentioned in the Introduction the photon counter responds always  $\frac{1}{c}r$  seconds later than the detector which signalsizes the appearance of the ground state  $X$ .



The source term of (7), however, cannot be neglected in this order. It indeed gives contribution to these singularities which will be included into the final formulae.

The amplitude  $M(r_1, r_2; t, \tau)$  calculated with the aid of the true initial amplitude (12) contains all the singularities which have been discussed above. The calculation whose main steps are summarized in the Appendix leads to the following result:

$$M(r_1, r_2; t, \tau) = T_{out}(r_2 - \tau) + T_{in}(r_2 - \tau) - T_{in}(-r_2 - \tau) + T_{sr}(r_2 - \tau) \quad (16)$$

where

$$\begin{aligned} T_{out}(\rho) &= \frac{Q_{ac}Q_{cb}}{2(2\pi)^4 r_1 r_2} e^{-i(\omega_{ba} - \frac{1}{2}\gamma)t} \cdot 2\pi i \left\{ e^{i(\omega_{ca} - \frac{1}{2}\gamma)r_1} e^{-i\omega_{cb}\rho} \right. \\ &\quad \cdot [Ei(i\omega_{cb}(\rho - r_1)) - Ei(\omega_{ca} - \frac{1}{2}\gamma)(\rho - r_1)] - \\ &\quad - \frac{1}{r_1 - \rho} \left[ \frac{1}{i(\omega_{ca} - \frac{1}{2}\gamma)} e^{i(\omega_{ba} - \frac{1}{2}\gamma)\rho} + \frac{1}{i\omega_{cb}} e^{i(\omega_{ba} - \frac{1}{2}\gamma)r_1} - \right. \\ &\quad \left. \left. - \frac{\omega_{cb} + \omega_{ca} - \frac{1}{2}\gamma}{i\omega_{cb}(\omega_{ca} - \frac{1}{2}\gamma)} \theta(\rho)\theta(t - \rho) e^{i(\omega_{ba} - \frac{1}{2}\gamma)\rho} \right] \right\} \\ T_{in}(\rho) &= \frac{Q_{ac}Q_{cb}}{2(2\pi)^4 r_1 r_2} e^{-i(\omega_{ba} - \frac{1}{2}\gamma)t} \cdot 2\pi i \left\{ e^{-i(\omega_{ca} - \frac{1}{2}\gamma)\rho} e^{-i\omega_{cb}r_1} \right. \\ &\quad \cdot [Ei(i(\omega_{ca} - \frac{1}{2}\gamma)(r_1 + \rho)) - Ei(i\omega_{cb}(r_1 + \rho))] + \\ &\quad \left. + \frac{1}{r_1 + \rho} \cdot \frac{1}{i\omega_{cb}} \left[ e^{-i(\omega_{ba} - \frac{1}{2}\gamma)\rho} - e^{i(\omega_{ba} - \frac{1}{2}\gamma)r_1} \right] \right\} \\ T_{sr}(\rho) &= \frac{Q_{ac}Q_{cb}}{2(2\pi)^4 r_1^2 r_2} e^{-i(\omega_{ba} - \frac{1}{2}\gamma)t} \left\{ - \frac{\omega_{cb} + \omega_{ca} - \frac{1}{2}\gamma}{i\omega_{cb}(\omega_{ca} - \frac{1}{2}\gamma)} \left[ 2e^{i(\omega_{ba} - \frac{1}{2}\gamma)r_1} \ln \frac{r_1}{|\rho|} + \right. \right. \\ &\quad \left. \left. + e^{i(\omega_{ba} - \frac{1}{2}\gamma)\rho} Ei(-i(\omega_{ba} - \frac{1}{2}\gamma)\rho) \right] + \frac{1}{i(\omega_{ca} - \frac{1}{2}\gamma)} e^{-i\omega_{cb}\rho} Ei(i\omega_{cb}\rho) - \right. \\ &\quad \left. - \frac{1}{2i\omega_{cb}} e^{i(\omega_{ca} - \frac{1}{2}\gamma_c)\rho} Ei(-i(\omega_{ca} - \frac{1}{2}\gamma_c)\rho) \right\} \end{aligned}$$

These amplitudes describe the contribution of the outgoing component, of the ingoing component and the influence of the source reduction respectively. The last term of  $T_{sr}(\rho)$  comes from the source of (6). The amplitudes are



important in those domains where the argument of the exponential integrals or logarithms is small. For  $(r_2 - r_1)\omega \gg 1$  the third, for  $(r_1 - r_2)\omega \gg 1$  the first term of (16) is negligible.

In order to have a crude estimate of the relative importance of the ingoing and outgoing components we compare their amplitudes at the point where the argument of their exponential integrals vanishes:

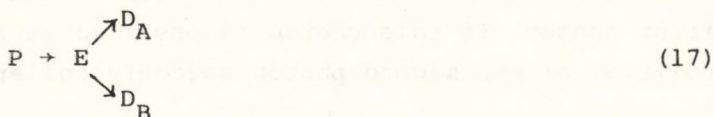
$$\left| \frac{T_{in}(-r_1)}{T_{out}(r_1)} \right| = \frac{1 - \frac{\ln \xi}{\xi - 1}}{1 + \frac{\ln \xi}{\xi - 1}}$$

As before  $\xi = \frac{\omega_{ca}}{\omega_{cb}} \geq 1$ . Since in this region  $\frac{\ln \xi}{\xi - 1}$  decreases monotonically from unity to zero the relative importance of the ingoing component decreases with decreasing  $\xi$  i.e. with the increasing excitation energy of the virtual state  $\phi_c$ .

Let us try to apply the ideas brought out by the above analysis to positronium annihilation. Since in this case the linear momentum is conserved the distribution function  $|M(\underline{r}_1, \underline{r}_2; t, 0)|^2$  at fixed  $\underline{r}_1$  is expected to be peaked at around  $\underline{r}_2 = -\underline{r}_1$ . The further time development of this distribution constitutes a rather difficult diffraction problem subjected to non-classical boundary conditions. Presumably, the field will be concentrated along the axis connecting the point  $\underline{r}_1$  with the origin. If  $r_A = r_B$  the coincidence spectrum will contain a single peak at  $\tau = 0$ . No anomalous peak at  $\tau = r_A + r_B$  arises. When  $r_A < r_B$  we have the usual peak at  $\tau = r_B - r_A$  if  $r_1 = r_A$  and an unusual one at the same value of  $\tau$  may also appear which corresponds to  $r_1 = r_B$ .

## 6. CONCLUSION

In a two-photon decay the events  $D_A$  and  $D_B$  associated with the responses of the counters may be space-like with respect to each other. Experience tells us that in such situations we can find always an event  $E$  which is the common cause and which in the present case is the emission event. Therefore, the causal chain of events can be represented by the diagram



where  $P$  is the decaying state preparation event. On the other hand, this organisation of events is compatible only with the simple time correlation, consisting of a single peak at  $\tau = r_B - r_A$  ( $r_A \leq r_B$ ).

The considerations of the preceding section suggest that quantum theory leads to a considerably more complicated correlation curve, consisting of several peaks, only one of which is compatible with the scheme (17). If, for example, one tries to reconcile the peak at  $\tau = r_B + r_A$  with this scheme one has



to admit that on one of the links  $E \rightarrow D_A$  and  $E \rightarrow D_B$  the arrow points backward in time i.e. the effect preceds the cause.

Let us return now to the question of reality of E raised in the Introduction. If in the interference experiment discussed there a two-photon emitter is used as the source one will not observe interference between the parts of the wave emitted at different moments of time i.e. the coherence length turns out to be much smaller than  $\gamma^{-1}$ . The reason is that the single-photon spectrum is continuous in the interval  $(0, \omega_{ba})$  and, according to the time-energy uncertainty relation which now can be safely applied the coherence length\* is  $\omega_{ba}^{-1}$ . It can be noted that this coherence length is equal to the range in  $r'_2 - r_2$  of the function

$$\int_0^\infty dt \int_0^\infty dr_1 \int_0^\infty dr'_2 M(r_1, r'_2; t, 0) M(r_1, r_2; t, 0).$$

Though the smallness of the range of  $M(r_1, r_2; t, 0)$  does not follow rigorously from the smallness of the range of this function the value of the single-photon coherence length seems to offer the real explanation of why the two-photon correlation found in Section 4 is as narrow as it is.

We see that in a two-photon time correlation experiment the emission event E does not lose its reality\*\* in the same sense as it does in the interference experiment discussed in the Introduction. But E seems to be deprived definitely of its function of being the common cause of the detectors' response which was in fact the only source of its heuristical significance for a time correlation experiment. The results of the preceding section are better represented by a causal chain  $P \rightarrow E \rightarrow D_A \rightarrow D_B$  or  $P \rightarrow E \rightarrow D_B \rightarrow D_A$ , depending on whether the counter A or B operates first. These two schemes can be replaced by

$$P \rightarrow E \rightarrow D_1 \rightarrow D_2 \quad (18)$$

whose last link indicates that the detection event which takes place first is at the same time the cause of the second detection event. The calculations show clearly that the elimination of the common cause is possible only due to the reduction of the wave-function at the moment  $t$  of the detection of the first photon. If this photon is observed at a distance  $r_1$  from the source the position of the second photon suddenly collapses onto a whole sphere of radius

\*In positronium annihilation this quantity is determined by the parameter, characterizing the degree of localization of the center of mass.

\*\*This does not mean that there cannot be conceived experimental situations in which the reality of E is lost even in a two-photon decay.



$r_1$  and this instantaneous change makes it possible to reconcile the third link of (18) with the possible space-like character of  $D_A$  and  $D_B$ .

This picture, including source reduction, does not contradict relativistic causality with respect to the event P since both  $D_A$  and  $D_B$  are inside the future light cone of P. Therefore, since of the three events only P is under human control, the relativistic acausality of the state reduction at  $t$  permits no faster-than-light transmission of information. The results of the calculation do not contradict available experimental evidence [2] either since the resolution of the coincidence spectra obtained so far is not high enough to resolve peaks of the type discussed in the preceding section.

Let us summarize in a qualitative way what has been found. The continuous character of the single-photon spectrum manifests itself in the strong radial correlation of photons. As a consequence of this strong correlation the photon field appears concentrated in a narrow spherical shell at the moment of the first detector response. Further time development of this distribution proceeds in the form of an outgoing and of an ingoing spherical wave. Since the initial distribution is strongly limited in radial direction it is unlikely that the ingoing component is effectively suppressed. The outgoing wave gives rise to the expected peak of the coincidence curve, but the ingoing wave leads to additional coincidences of unusual properties. In particular it leads to a doublet at  $\tau=r_A+r_B$  which looks like as if one of the photons propagated backward in time. This component does not appear in the coincidence curve of positronium annihilation and  $\pi^0 \rightarrow 2\gamma$  decay. The collapse of the source state leads to further structure. All these peculiarities are direct consequences of the assumed instantaneous nature of the wave-function collapse at the moment of an observation.

## APPENDIX

As discussed in the Introduction terms of the relative order  $(\omega L)^{-1}$  will be systematically omitted. Here  $\omega$  is any of the Bohr-frequencies  $\omega_{ba}, \omega_{ca}, \omega_{cb}$  and  $L$  is any of the macroscopic distances  $r_A, r_B, t-r_A, t-r_B$ .

The domain of integration in (10) will be divided into three parts

$$I_a: 0 \leq r'_2 \leq R_1$$

$$I_b: R_1 \leq r'_2 \leq R_2$$

$$I_c: R_2 \leq r'_2$$

with  $R_1$  and  $R_2$  introduced in Section 4. We find approximate expressions  $M_i(r_1, r_2; t, 0)$  ( $i=a, b, c$ ) whose domains of validity overlap in the neighborhood of  $R_1$  and  $R_2$ . Therefore, when the three integrals are added up the parameters  $R_1$  and  $R_2$  drop out of the sum in every order of  $(\omega L)^{-1}$  where  $L$  may represent  $R_1, r_1 - R_1, R_2 - r_1, t - R_2$  also.

In the region  $I_a$  the exponential integrals of (12) can be replaced by their asymptotic expressions and for  $M_a$  we obtain a combination of integrals of the form



$$\int_0^{\infty} du \frac{e^{-i\alpha u}}{u - \omega_{ba} + \frac{i}{2}\gamma} \quad (A1)$$

and

$$\int_0^{\infty} du \frac{e^{-i\alpha u}}{\omega + \omega_{cb}} \quad (A2)$$

When  $\alpha$  is of the type L /in this region  $t-r'_2$  is also of this type/ (A2) can be dropped and

$$\int_0^{\infty} du \frac{e^{-iLu}}{u - \omega_{ba} + \frac{i}{2}\gamma} = (-2\pi i) e^{-iL(\omega_{ba} - \frac{i}{2}\gamma)} \theta(L) + O\left(\frac{1}{\omega L}\right),$$

where  $\theta(x)$  is the step-function. If  $\alpha = \pm r'_2$  (A1) and (A2) can be expressed in terms of exponential integrals. As a result we obtain

$$\begin{aligned} M_a(r_1, r'_2; t, 0) &= \frac{Q_{ac} Q_{cb}}{2(2\pi)^4 r_1 r'_2} \cdot \frac{(-2\pi i)}{i(r_1 - r'_2)} e^{-i(\omega_{ba} - \frac{i}{2}\gamma)t} \\ &\left\{ \frac{\omega_{cb} + \omega_{ca} - \frac{i}{2}\gamma}{\omega_{cb}(\omega_{ca} - \frac{i}{2}\gamma)} (e^{i(\omega_{ba} - \frac{i}{2}\gamma)r_1} - e^{i(\omega_{ba} - \frac{i}{2}\gamma)r'_2}) + \right. \\ &\left. + \frac{1}{2\pi i} \frac{\omega_{cb} + \omega_{ca} - \frac{i}{2}\gamma}{\omega_{cb}(\omega_{ca} - \frac{i}{2}\gamma)} H(r'_2, \omega_{ba} - \frac{i}{2}\gamma) + \frac{1}{2\pi i} \frac{1}{\omega_{ca} - \frac{i}{2}\gamma} H(r'_2, \omega_{cb}) \right\} \end{aligned}$$

where

$$H(x, \Omega) = e^{i\Omega x} \text{Ei}(-i\Omega x) - e^{-i\Omega x} \text{Ei}(i\Omega x). \quad (A3)$$

In particular

$$\begin{aligned} r_2 M_a(r_1, r_2; t, 0) \Big|_{r_2=0} &= \frac{Q_{ac} Q_{cb}}{2(2\pi)^4 r_1} \cdot \frac{(-2\pi i)}{i\omega_{cb} r_1} e^{-i(\omega_{ba} - \frac{i}{2}\gamma)t} \\ &\cdot \left[ \frac{\omega_{cb} + \omega_{ca} - \frac{i}{2}\gamma}{\omega_{ca} - \frac{i}{2}\gamma} e^{i(\omega_{ba} - \frac{i}{2}\gamma)r_1} - \frac{1}{2} \right] \neq 0. \end{aligned}$$



In the region  $I_b$  we have to calculate integrals of the type

$$\int_0^{\infty} du \frac{e^{-i\alpha u} \text{Ei}(i(r'_2 - r_1)(\omega_{cb} + u))}{u - \omega_{ba} + \frac{i}{2}\gamma} =$$

$$= (-2\pi i) \theta(\alpha) e^{-i(\omega_{ba} - \frac{i}{2}\gamma)\alpha} \text{Ei}(i(r'_2 - r_1)(\omega_{ca} - \frac{i}{2}\gamma)) + O(\frac{1}{\omega L}).$$

In writing down the right hand side we have taken into account that  $\alpha$  is of the type L /now  $\pm r'_2$  and  $t - r'_2$  are of this type also/ and that in the important region  $r_1 \approx r'_2$  of the exponential integral  $-\alpha + r'_2 - r_1$  is of the same sign as  $\alpha$ . Closer examination reveals correction terms of the type  $(\omega L)^{-1} \text{Ei}(i(r'_2 - r_1)\omega_{cb})$  which at  $r_1 \approx r'_2$  are not small. However, when integrated over  $r'_2$  these terms give negligible contribution to  $M(r_1, r_2; t, \tau)$  of the order of  $(\omega L)^{-1}$ . The expression  $M_b(r_1, r_2; t, 0)$  is given in (14).

In the region  $I_c$  we have again integrals of the form (A1) and (A2) where the  $\alpha$ -s, except  $t - r'_2$ , are of the type L. We obtain

$$M_c(r_1, r'_2; t, 0) = \frac{Q_{ac} Q_{cb}}{2(2\pi)^4 r_1 r_2} \frac{(-2\pi i)}{i(r_1 - r'_2)} e^{-i(\omega_{ba} - \frac{i}{2}\gamma)t}$$

$$\cdot \left[ \frac{\omega_{cb} + \omega_{ca} - \frac{i}{2}\gamma}{\omega_{cb}(\omega_{ca} - \frac{i}{2}\gamma)} (e^{i(\omega_{ba} - \frac{i}{2}\gamma)r_1} - \theta(t - r'_2) e^{i(\omega_{ba} - \frac{i}{2}\gamma)r'_2}) \right] +$$

$$+ \frac{Q_{ac} Q_{cb}}{2(2\pi)^4 r_1 r_2} \frac{1}{i(r_1 - r'_2)} \left[ \frac{\omega_{cb} + \omega_{ca} - \frac{i}{2}\gamma}{\omega_{cb}(\omega_{ca} - \frac{i}{2}\gamma)} e^{-i(\omega_{ba} - \frac{i}{2}\gamma)(t - r'_2)} \text{Ei}(i(\omega_{ba} - \frac{i}{2}\gamma)(t - r'_2)) - \right.$$

$$\left. - \frac{1}{\omega_{ca} - \frac{i}{2}\gamma} e^{i\omega_{cb}(t - r'_2)} \text{Ei}(-i\omega_{cb}(t - r'_2)) \right].$$

The  $r'_2$ -integration in (10) can be expressed as a sum of integrals which are of the same structure the prototype of which is\*

$$K = \int_0^{R_2} dr'_2 \left[ \frac{1}{r'_2 - \rho - i\epsilon} + \frac{1}{r'_2 + \rho + i\epsilon} \right] e^{i\Omega_1 r'_2} \cdot \text{Ei}(i\Omega_2(r_1 - r'_2)),$$

where  $\rho$  is a real number and  $\Omega_1 = \pm \omega_{cb}$  or  $\pm(\omega_{ca} - \frac{i}{2}\gamma)$ .

\*For some parts of M it is possible to integrate over the combined region  $I_a + I_b$ . K is an integral of this type.



In the  $r'_2$  plane the exponential integral has a cut

$$r'_2 = r_1 + i\Omega_2\eta + i\varepsilon \cdot \text{Re}\Omega_2 \quad ; \quad \eta \geq 0.$$

If  $\text{Re}\Omega_2 < 0$  ( $\text{Re}\Omega_2 > 0$ ) the cut lies in the lower (upper) half-plane. Since the distance of the branch-point from the integration limits is of the type L we can deform the contour of integration into a new contour C along which the exponential integral can be replaced by the first term of its asymptotic expansion. To meet this requirement C must not cross the cut i.e. it must lie in the upper (lower) half-plane when  $\text{Re}\Omega_2 < 0$  ( $\text{Re}\Omega_2 > 0$ ). We will choose it in such a manner as to make the pole at  $r'_2 = \rho + i\varepsilon$  ( $r'_2 = -\rho - i\varepsilon$ ) enclosed between C and the  $(0, R_2)$  interval of the real axis. After having replaced the exponential integral on C by its asymptotic value, we deform C back to the real axis. As a result we obtain the difference of the values of the exponential integral and of its asymptotic form at the pole and a new integral which can be expressed through logarithms or exponential integrals. When, for example  $\text{Re}\Omega_2 < 0$  we have

$$K = \int_0^{R_2} dr'_2 \left[ \frac{1}{r'_2 - \rho - i\varepsilon} + \frac{1}{r'_2 + \rho + i\varepsilon} \right] e^{i\Omega_1 r'_2} \frac{e^{i\Omega_2 (r_1 - r'_2)}}{i\Omega_2 (r_1 - r'_2)} +$$

$$+ 2\pi i e^{i\Omega_1 \rho} \left[ \text{Ei}(i\Omega_2 (r_1 - \rho)) - \frac{e^{i\Omega_2 (r_1 - \rho)}}{i\Omega_2 (r_1 - \rho - i\varepsilon)} \right].$$

If  $\Omega_1 = \omega_2 = -\omega_{cb}$ , then

$$K = \frac{e^{-i\omega_{cb} r_1}}{-i\omega_{cb}} \left\{ \frac{1}{r_1 - \rho} \left[ \ln \left| \frac{r_1 (R_2 - \rho)}{\rho (R_2 - r)} \right| + \right. \right.$$

$$+ i\pi (\theta(\rho) \theta(R_2 - \rho) - 1) \left. \right] + \frac{1}{r_1 + \rho} \left[ \ln \left| \frac{r_1 (R_2 + \rho)}{\rho (R_2 - r_1)} \right| + \right.$$

$$\left. + i\pi (1 - \theta(-\rho) \theta(R_2 + \rho)) \right] \left. \right\} + 2\pi i e^{-i\omega_{cb} \rho} \text{Ei}(-i\omega_{cb} (r_1 - \rho)).$$

In  $I_b$  and  $I_c$  all the integrals are such that the distance of the branch-point from the integration limits is of the order of L and they can be handled in the same way as K. In  $I_a$ , however, beside this type we encounter also integrals in which the branch-point and the lower limit coincide. These integrals are of the form

$$L = \int_0^{R_1} dr'_2 \left[ \frac{1}{r'_2 - \rho - i\varepsilon} + \frac{1}{r'_2 + \rho + i\varepsilon} \right] \frac{1}{r_1 - r'_2} H(r'_2, \Omega)$$



where the function  $H$  is given by (A3). The integral  $L$  is appreciable only at  $\rho \approx 0$  and may be approximated by

$$\begin{aligned} L &= \frac{1}{r_1} \int_0^\infty dr'_2 \left[ \frac{1}{r'_2 - \rho - i\epsilon} + \frac{1}{r'_2 + \rho + i\epsilon} \right] H(r'_2, \Omega) = \\ &= \frac{1}{r_1} \int_0^\infty dr'_2 \left[ \frac{1}{r'_2 - \rho - i\epsilon} e^{i\Omega r'_2} \text{Ei}(-i\Omega r'_2) - \frac{1}{r'_2 + \rho + i\epsilon} e^{-i\Omega r'_2} \text{Ei}(i\Omega r'_2) \right] + \\ &+ \frac{1}{r_1} \int_0^\infty dr'_2 \left[ \frac{1}{r'_2 + \rho + i\epsilon} e^{i\Omega r'_2} \text{Ei}(-i\Omega r'_2) - \frac{1}{r'_2 - \rho - i\epsilon} e^{-i\Omega r'_2} \text{Ei}(i\Omega r'_2) \right]. \end{aligned}$$

Let us consider now the integral

$$\int_C dz \frac{e^{-\Omega z} \text{Ei}(\Omega z)}{z + ib} = \begin{cases} 2\pi i e^{i\Omega b} \text{Ei}(-i\Omega b) & \text{if } \text{Im}b < 0 \\ 0 & \text{if } \text{Im}b > 0 \end{cases}$$

where  $C$  consists of the imaginary axis and the left-side semicircle. This integral may be rewritten as

$$\begin{aligned} \int_0^\infty dy \left( \frac{1}{y-b} e^{i\Omega y} \text{Ei}(-i\Omega y) - \frac{1}{y+b} e^{-i\Omega y} \text{Ei}(i\Omega y) \right) &= \\ &= \begin{cases} -2\pi e^{i\Omega b} \text{Ei}(-i\Omega b) & \text{if } \text{Im}b < 0 \\ 0 & \text{if } \text{Im}b > 0 \end{cases} \end{aligned}$$

The integrals in  $L$  are of this type, hence

$$L = -\frac{2\pi i}{r_1} e^{-i\Omega \rho} \text{Ei}(i\Omega \rho).$$



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